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1986 J. Phys. A: Math. Gen. 19 1041

(http://iopscience.iop.org/0305-4470/19/6/031)

J. Phys. A: Math. Gen. 19 (1986) 1041-1048. Printed in Great Britain

Dynamic critical behaviour of semi-infinite systems: conformal invariance and mirror theory

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Received 24 June 1985, in final form 29 August 1985

Abstract. The critical dynamics of the d-dimensional semi-infinite systems is studied at bulk critical temperature. As a result of the conformal invariance, the dynamic response and correlation functions depend only on the 'real' distance r, the 'image' distance $\bar{r}$ and the time t (mirror theory). Assuming the non-conserved order parameter with $O(n)$ symmetry, we evaluate the response and correlation functions exactly in the large-n limit and also to first order in $\varepsilon = 4-d$. Our results satisfy the mirror theory for the various types of the phase transitions including the ordinary, special, anisotropic special and extraordinary transitions.

1. Introduction

Critical phenomena at surfaces have been studied extensively with considerable current interest (Binder 1983). For the static case, the scaling relations among surface critical exponents were investigated by use of several theoretical techniques (Reeve and Guttmann 1981, Diehl and Dietrich 1981a, b, Ohno and Okabe 1983a, b, c, 1984, Ohno *et al* 1984) and the various aspects of phase transitions aroused a lot of experimental interest (Alvarado *et al* 1982, Gidley *et al* 1982, Schlossman *et al* 1985, Weller *et al* 1985). The general surface equation of state was also discussed recently (Nakanishi and Fisher 1982, Okabe and Ohno 1984, Kikuchi and Okabe 1985a, b).

In the series of papers on the $1/n$ expansion and $\varepsilon(=4-d)$ expansion, Ohno and Okabe (1983c, 1984, 1985) pointed out that the mirror theory holds in the semi-infinite systems at bulk criticality; the two-point correlation function $G(\boldsymbol{r}_1, \boldsymbol{r}_2)$ of the semi-infinite systems depends only on the 'real' distance $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ and the 'image' distance $\bar{r} = |\boldsymbol{r}_1 - \boldsymbol{r}_2^*|$ ($\boldsymbol{r}_2^*$ is the image point of $\boldsymbol{r}_2$) at bulk critical temperature. This mirror theory automatically results in the surface scaling relation $2\eta_\perp - \eta_\parallel = \eta$.

Recently, Cardy (1984) showed that the special property of $G(\boldsymbol{r}_1, \boldsymbol{r}_2)$ (mirror theory) can be attributed to the invariance under the conformal transformations which preserve the surface geometry (see also Ohno and Okabe 1985). Conformal invariance is the idea originally introduced to the critical phenomena by Polyakov (1970). The fixed-point Hamiltonian and hence the many-point correlation functions at criticality are expected to be invariant under the local renormalisation group (RG) transformations which consist of the following two steps. The first step is the conformal transformation corresponding locally to a dilatation,

$$\boldsymbol{r}_j' = b(\boldsymbol{r}_j)^{-1}\boldsymbol{r}_j, \tag{1.1}$$

plus a rotation. The second step is a local rescaling of each spin variable with a factor $b(\boldsymbol{r}_j)^{-(d-2+\eta)/2}$ (η is the anomalous dimension of the spin). The constraint due to the conformal invariance becomes stronger in two-dimensional ($d=2$) systems and determines the explicit form of the correlation functions (Cardy 1984, Burkhardt 1985).

The dynamics of the semi-infinite systems, on the other hand, has been studied by several authors. The relaxation process near surfaces was first discussed by Kumar (1974) and Kumar and Maki (1976). The finite-size scaling study of the time-dependent Ginzburg-Landau (TDGL) model was elaborated by Suzuki (1977). Tanaka (1979) dealt with the semi-infinite chain system exactly. More recently, Dietrich and Diehl (1983) carried through the field-theoretical ε expansion for the dynamics of the semi-infinite TDGL model and asserted that there is no genuine surface dynamic exponent. The general theory concerning this property was explored by Kikuchi and Okabe (1985c) with the use of Monte Carlo simulation.

The purpose of this paper is to show that the conformal invariance and the mirror theory hold for the dynamics of the general semi-infinite systems at bulk criticality. This is the *first* report to treat the conformal invariance of the semi-infinite systems as a dynamic problem. We will discuss this subject in the following ways. In § 2, we derive the mirror theory for the dynamics by assuming the conformal invariance. Next, developing the exact $n\to\infty$ solution (§ 3) and the expansion in $\varepsilon=4-d$ (§ 4), we explicitly obtain the response and correlation functions which satisfy the mirror theory. Discussions are given in § 5.

2. Mirror theory for dynamics

The idea of the conformal invariance can be extended to the dynamics†. It is convenient to introduce different times $t_1, t_2, \ldots, t_N$ for individual particles‡. Then the dynamic correlation functions at criticality are invariant under the RG transformations accompanied by local time rescalings $t_j'=b(\boldsymbol{r}_j)^{-z}t_j$ with the dynamic exponent z.

The invariance under the special conformal transformation

$$\frac{\boldsymbol{r}_j'}{r_j'^2}=\frac{\boldsymbol{r}_j}{r_j^2}+\boldsymbol{a} \tag{2.1}$$

which does not change the surface geometry ($\boldsymbol{a}$ is parallel to the surface plane) permits only two independent space variables in the static correlation function. They are the real distance r and the image distance $\bar{r}$ (Ohno and Okabe 1983c, 1985). It should be noticed that there is the same property in the case of the dynamics because we may expect the invariance under the same conformal groups. Hence, together with the requirements of homogeneous functions, the dynamic response and correlation functions of the semi-infinite systems at $T=T_c$ are written as

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2; t)=G^{\text{bulk}}(r; t)\Theta(r/\bar{r}, t\zeta(r)), \tag{2.2a}$$

$$C(\boldsymbol{r}_1, \boldsymbol{r}_2; t)=C^{\text{bulk}}(r; t)\Phi(r/\bar{r}, t\zeta(r)), \tag{2.2b}$$

respectively. Here $G^{\text{bulk}}(r; t)$ and $C^{\text{bulk}}(r; t)$ are the response and correlation functions of the corresponding bulk system. The function $\zeta(r)$, whose dimensionality is equal

† The bulk $d=2$ system was discussed quite recently by Cardy (1986).
‡ From another viewpoint, Abe (1984) exploited this formalism to obtain an exact closed equation of the static correlation function.

to that of the frequency ω, specifies the scaling law between the real distance r and the time interval t, so that it agrees with the bulk one

$$\zeta(r) = \Gamma r^{-z}. \tag{2.3}$$

Here z is the usual bulk dynamic exponent (Ma 1976, Hohenberg and Halperin 1977). Equation (2.2) with (2.3) gives the mirror theory for the dynamic problem. This equation can be recast in various ways. The convenient one is

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2; t) = G_\rho^t(y_1, y_2) = (y_1 y_2)^{(2-d-\eta-z)/2} g(v, \tau), \tag{2.4a}$$

$$C(\boldsymbol{r}_1, \boldsymbol{r}_2; t) = C_\rho^t(y_1, y_2) = (y_1 y_2)^{(2-d-\eta)/2} f(v, \tau). \tag{2.4b}$$

Here y_1 and y_2 are the normal distances from the surface (we use y_j instead of the conventional z_j to avoid any confusion with the exponent z), ρ is the distance parallel to the surface and v and τ denote

$$v = \frac{y_1^2 + y_2^2 + \rho^2}{2 y_1 y_2} = \frac{\bar{r}^2 + r^2}{\bar{r}^2 - r^2} \tag{2.5}$$

and

$$\tau = \frac{2(\Gamma t)^{2/z}}{y_1 y_2}, \tag{2.6}$$

respectively. The important point of these equations is that the timescale is determined by the bulk dynamic exponent. This point is to be identified with the conclusion of the paper of Dietrich and Diehl (1983) and with the concept used by Suzuki (1977). However, the equations from (2.2) to (2.6) impose a stronger restriction on the response and correlation functions in real space because only v and τ are allowed for their arguments.

Once the mirror theory is assumed, it is easy to derive various asymptotic forms. Let us focus our attention on the correlation function. First of all, in the limit $r/\bar{r} \to 0$ or equivalently in the limit $v \to 1$, $\Phi(r/\bar{r}, t\zeta(r))$ in (2.2) becomes unity since it approaches the bulk correlation function. On the other hand, when $y_1 \sim 0$, the function $f(v, \tau)$ in (2.4b) behaves as $v^{-\tilde{\eta}} \cdot \hat{f}(v/\tau)$. Here the function $\hat{f}(v/\tau)$ becomes constant in the limit $v/\tau \to \infty$; $\tilde{\eta}$ is the static exponent which is related to $\eta_\parallel$ and $\eta_\perp$ via $\eta_\parallel = 2 - d + 2\tilde{\eta}$ and $\eta_\perp = (2 - d + \eta)/2 + \tilde{\eta}$ (Ohno and Okabe 1985). Thus we have

$$\begin{aligned} C_\rho^t(y_1, y_2) &\xrightarrow[y_1 \sim 0, \rho \to \infty]{} \rho^{2-d-\eta_\parallel} \\ &\xrightarrow[y_1 \sim 0, y_2 \to \infty]{} y_2^{2-d-\eta_\perp} \\ &\xrightarrow[y_1 \sim 0, t \to \infty]{} \tau^{-\tilde{\eta}} \simeq t^{(2-d-\eta_\parallel)/z}. \end{aligned} \tag{2.7}$$

In deriving the last relation in (2.7), we used the result of the short distance expansion showing that $\hat{f}(v/\tau)$ behaves as $(v/\tau)^{\tilde{\eta}}$ for $v/\tau \to 0$ (Dietrich and Diehl 1983). Of course, we retrieve the static correlation function if we put $t = 0$ in $C_\rho^t(y_1, y_2)$.

3. Large-n limit

Below, we will show that the response and correlation functions have indeed the forms like (2.4a) and (2.4b). Specifically, we will consider the case of the non-conserved order parameter (case A of Hohenberg and Halperin 1977) with $O(n)$ symmetry and omit any mode-mode coupling terms. Our calculation here is based on the following two limiting approaches. One is the exact approach in the large-n limit and the other is the first-order (one-loop) approximation in the $\varepsilon(=4-d)$ expansion. Since the latter can be performed along the same line as the former, we will mainly concentrate on the large-n limit. The points to be modified in the case of the ε expansion will be mentioned briefly in the next section.

We start from the semi-infinite TDGL model at $T=T_c$ associated with the Hamiltonian

$$H=\int \mathrm{d}^{d-1}\rho\left\{\int_0^\infty \mathrm{d}y\left[\sum_{\alpha=1}^{n}\frac{1}{2}(\nabla\phi_\alpha)^2+\frac{u}{4}\left(\sum_{\alpha=1}^{n}\phi_\alpha^2\right)^2\right]\right.$$
$$\left.+\left(\frac{c_e}{2}\sum_{\alpha=1}^{m}\phi_\alpha^2+\frac{c_h}{2}\sum_{\alpha=m+1}^{n}\phi_\alpha^2\right)_{y=0}\right\}. \tag{3.1}$$

We include a possibility of the surface anisotropy and take for granted that $c_e \leqslant c_h$ (Diehl and Eisenriegler 1984, Ohno *et al* 1985). The surface order exists when c_e is lower than the special critical value c^* which is known to be $O(\varepsilon)$. When $c_e > c^*$, the ordinary transition (O) is expected generally, whereas the special transition (Sp) occurs for $c_e = c_h = c^*$. If $c_e = c^*$ and $c_h > c^*$, the anisotropic special transition (ASp) takes place. Otherwise, if $c_e < c^*$, the surface order lives at bulk T_c where the extraordinary transition (E) takes place. The schematic phase diagram is shown in figure 1. Following the standard derivation (see, for example, Ma 1976), we find that the response function obeys

$$\left(\frac{1}{\Gamma}\frac{\partial}{\partial t}-\nabla_\rho^2-\frac{\partial^2}{\partial y_1^2}+\frac{\mu^2-\frac{1}{4}}{y_1^2}\right)G(\boldsymbol{r}_1,\boldsymbol{r}_2;t)=\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\delta(t) \tag{3.2}$$

in the large-n limit. Here the term proportional to y_1^{-2} arises from the self-energy $\langle\phi_\alpha^2\rangle$, which is equal to that evaluated for the statics. For the discussions in the case of the

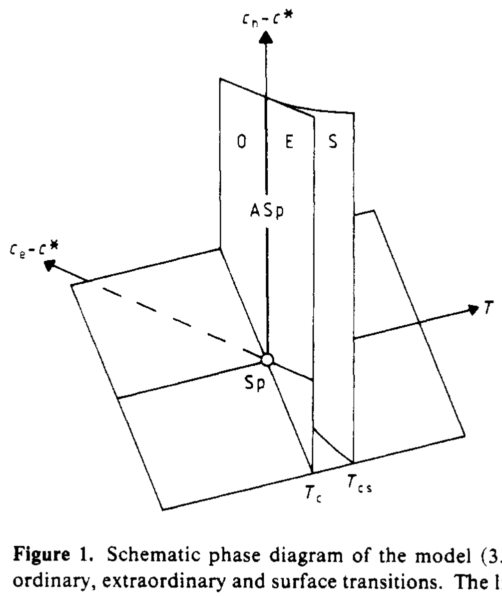

Figure 1. Schematic phase diagram of the model (3.1). Planes, O, E and S indicate the ordinary, extraordinary and surface transitions. The line of intersection ASp and the small circle Sp correspond to the anisotropic and isotropic special transitions, respectively.

statics, see Bray and Moore (1977), Ohno and Okabe (1983a, b, c, 1984) and Ohno *et al* (1985). The values of the parameter μ are listed in table 1.

Since there is a translational symmetry parallel to the surface, we may transform (3.2) into the $q\omega$ space and have

$$\left(q^2-\frac{\mathrm{i}\omega}{\Gamma}-\frac{\partial^2}{\partial y_1^2}+\frac{\mu^2-\frac{1}{4}}{y_1^2}\right)G_q^\omega(y_1,y_2)=\delta(y_1-y_2). \tag{3.3}$$

The solution of (3.3) is readily obtained by using the Fourier-Bessel transformation

$$G_q^\omega(y_1,y_2)=\int_0^\infty \frac{\mathrm{d}s\, s}{s^2+q^2-\mathrm{i}\omega/\Gamma}J_\mu(sy_1)J_\mu(sy_2). \tag{3.4}$$

One can easily check that $G_q^{\omega\to 0}(z,z')$ reduces to the static correlation function. Then the inverse transformation can be easily performed by noting

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}\omega t)}{s^2+q^2-\mathrm{i}\omega/\Gamma}\mathrm{d}\omega=\Gamma\theta(t)\exp[-\Gamma(s^2+q^2)t], \tag{3.5}$$

$$\int \mathrm{d}^{d-1}q\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{\rho})\exp(-\Gamma q^2 t)=\frac{1}{(4\pi\Gamma t)^{(d-1)/2}}\exp\left(-\frac{\rho^2}{4\Gamma t}\right). \tag{3.6}$$

In equation (3.5), $\theta(t)$ denotes the Heaviside step function. The rest integral may be carried out by the integral formula (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty \mathrm{d}s\, s\exp(-\Gamma s^2 t)J_\mu(sy_1)J_\mu(sy_2)$$
$$=\frac{1}{2\Gamma t}\exp\left(-\frac{y_1^2+y_2^2}{4\Gamma t}\right)I_\mu\left(\frac{y_1y_2}{2\Gamma t}\right) \tag{3.7}$$

($I_\mu(x)$ is the modified Bessel function). Thus we finally obtain the real-space response

Table 1. List of the values of μ given in the large-n limit and in the ε expansion. The parameter μ, which is related to the surface critical exponent $\eta_\parallel$ via $\eta_\parallel=1+2\mu$, has different values according to the spatial dimension d, the spin dimension n and the type of phase transition. For the extraordinary transition, $\mu=(d-1)/2$ gives the *exact* surface critical exponents for the transverse component (Ohno and Okabe 1984).

Type of transition	Large-n limit		ε expansion
Ordinary	$(d-3)/2$	$(2<d<4)$	$\frac{1}{2}-\frac{n+2}{2(n+8)}\varepsilon$
Special	$(d-5)/2$	$(3<d<4)$	$-\frac{1}{2}-\frac{n+2}{2(n+8)}\varepsilon$
Anisotropic special (hard component)	$(d-3)/2$	$(3\leq d<4)$	$\frac{1}{2}-\frac{n-2m+2}{2(n+8)}\varepsilon$
Anisotropic special (easy component)	$(3-d)/2$	$(3\leq d<4)$	$-\frac{1}{2}+\frac{n-2m-2}{2(n+8)}\varepsilon$
Extraordinary (transverse component)	$(d-1)/2$	$(2<d<4)$	

function

$$G^t_\rho(y_1, y_2) = \frac{\Gamma\theta(t)}{(2\pi)^{(d-1)/2}} \frac{(y_1 y_2)^{1/2}}{(2\Gamma t)^{(d+1)/2}} \exp\left(-\frac{v}{\tau}\right) I_\mu\left(\frac{1}{\tau}\right) \tag{3.8}$$

with v and τ given by (2.5) and (2.6) (we must put $z=2$). Obviously the response function (3.8) has the form of (2.4a).

The correlation function is obtained by the fluctuation-dissipation theorem (Ma 1976)

$$C^\omega = (2/\omega) \operatorname{Im} G^\omega. \tag{3.9}$$

In the present case, we can simply rewrite the relation (3.9) in t space and have

$$C^t = \int_{|t|}^\infty \mathrm{d}t\, G^t. \tag{3.10}$$

Expanding (3.8) in the power series of $1/t$ and carrying out the t integration term by term, we see that C^t is also in the form of (2.4b). The long-distance or long-time behaviour is given by (2.7) with $\eta_\parallel = 1+2\mu$, $\eta=0$ and $z=2$ for each type of phase transition.

4. ε expansion

In the mean-field (ε^0) approximation (which is equivalent to taking the limit $d\to4$ in the expressions of $n\to\infty$ limit), we have equation (3.2) or (3.3) with $\mu=\pm\frac{1}{2}$. The dynamic correlation function beyond the mean-field theory was discussed by Dietrich and Diehl (1983) in Fourier space. To first order in ε, only the one-loop graph illustrated in figure 2 contributes to the response function whose self-energy is equal to that evaluated for the statics. It leads to a deviation of the value of μ from $\pm\frac{1}{2}$. Table 1 lists the resulting values of μ for various phase transitions (Lubensky and Rubin 1975, Bray and Moore 1977, Diehl and Eisenriegler 1984). All the other calculations from (3.2) to (3.10) are perfectly the same as before and we have the same response function (3.8) but with the different values of μ. Expansion of (3.8) around the mean-field value $\mu=\pm\frac{1}{2}$ may be obtained by utilising

$$\left(\frac{\partial I_\nu}{\partial \nu}\right)_{\nu=\pm\frac{1}{2}} = \frac{1}{(2\pi x)^{1/2}} (\mathrm{e}^x \operatorname{Ei}(-2x) \mp \mathrm{e}^{-x} \overline{\operatorname{Ei}}(2x)). \tag{4.1}$$

Here $\operatorname{Ei}(-2x)$ and $\overline{\operatorname{Ei}}(2x)$ are the exponential integral function and the related exponential integral function, respectively. Combining (3.8) with (4.1) and introducing a new

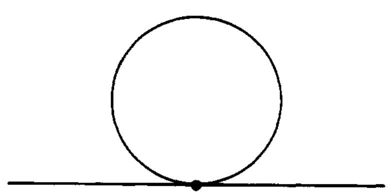

Figure 2. One-loop graph appearing in the ε expansion. The self-energy part of this graph has the same contribution as that given in the static calculations.

parameter κ by

$$\mu = \pm\tfrac{1}{2} + \kappa\varepsilon, \tag{4.2}$$

we have the response function accurate to $O(\varepsilon)$ as

$$G_p^t(y_1, y_2) = \frac{1}{(2\pi)^{d/2}} \frac{\Gamma\theta(t)}{(2\Gamma t)^{d/2}} \{\exp[-(v-1)/\tau][1+\kappa\varepsilon\,\mathrm{Ei}(-2/\tau)]$$
$$\mp \exp[-(v+1)/\tau][1+\kappa\varepsilon\,\overline{\mathrm{Ei}}(-2/\tau)]\} + O(\varepsilon^2). \tag{4.3}$$

The value of κ for each type of phase transition is found by the substitution of μ listed in table 1 into (4.2). The correlation function is obtained by the fluctuation-dissipation theorem which in the present case is equal to (3.10). The asymptotic behaviour of the correlation function is again given by (7) with $\eta_\parallel = 1+2\mu$, $\eta = 0$ and $z = 2$. This result is accurate to $O(\varepsilon)$.

5. Discussions

In this paper, we have discussed the conformal invariance and the mirror theory which is expected for the dynamics of the semi-infinite systems. One of the important conclusions of this paper is that the response and correlation functions at bulk critical temperature have the forms given by (2.4). We have called this special property the mirror theory because these functions depend on the space variables only through two combinations, i.e. the real distance r and the image distance $\bar{r}$. Such a property is well known in classical electrodynamics. Of course, our systems are more complicated due to their intrinsic non-linearity, so that the principle of superposition is invalid beyond the mean-field theory.

We have shown that the mirror theory is certainly satisfied in the two specific approaches: (i) the exact approach in the large-n limit; (ii) the one-loop approximation in the ε expansion. For both approaches the resulting response function in real space is given by (3.8) and the correlation function is expressed in an integral form (3.10). In the case of the ε expansion, an alternative form of (3.8) which is expanded to $O(\varepsilon)$ is given by (4.3). The values of the parameter μ are listed in table 1.

We have considered several possible phase transitions including the ordinary transition, the special transition, the anisotropic special transition and the extraordinary transition. The mirror theory should be valid for all of these transitions. In the case of the extraordinary transition, we have not discussed the longitudinal component of the correlation function. This is only due to the complexity. In our earlier work (Ohno and Okabe 1984), we have treated the static longitudinal function in the large-n limit and shown that it satisfies

$$C_p^{t=0}(y_1, y_2) = (y_1 y_2)^{1-d/2} f(v), \tag{5.1}$$

$$\left((v^2-1)\frac{d^2}{dv^2} + (1+2\mu)v\frac{d}{dv}\right)^2 f(v) = \tilde{A}(\mu)\left(\frac{Q_{2\mu}^2(v)}{v^2-1} + \frac{1+2\mu}{2\mu-3}\frac{Q_{-1+2\mu}^1(v)}{(v^2-1)^{1/2}}\right), \tag{5.2}$$

where $\tilde{A}(\mu)$ is a function of $\mu = (d-1)/2$ and $Q_\nu^\sigma(x)$ is the associated Legendre function of the second kind. Equation (5.1) is just the static limit of (2.3*b*), so that we might expect that the corresponding dynamic correlation function be given in the form of (2.4*b*).

Acknowledgments

We are very grateful to A Morita, K Niizeki and M Kikuchi for helpful discussions. We are also grateful to J L Cardy and T W Burkhardt for sending us preprints (Cardy 1985, Burkhardt 1985) prior to publication.

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